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On Critical Phenomena in a Noncommutative Space

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Abstract

In this paper we demonstrate that coordinate noncommutativity at short distances can show up in critical phenomena through UV-IR mixing. In the symmetric phase of the Landau-Ginsburg model, noncommutativity is shown to give rise to a non-zero anomalous dimension at one loop, and to cause instability towards a new phase at large noncommutativity. In particular, in less than four dimensions, the one-loop critical exponent η is non-vanishing at the Wilson-Fisher fixed point.

I. INTRODUCTION

Space quantization is an old idea. Specific coordinate commutation relations were suggested more than half century ago [1,2], in an attempt to resolve the short-distance singularities in local quantum field theory. Recently, interests in field theories on a noncommutative space (or simply NCFT) have been revived [3], particularly because of their emergence in string/M(atrix) theory [4,5]. Also NCFT is expected to be relevant to planar quantum Hall systems in a strong magnetic field, since the guiding-center coordinates for the cyclotron motion of a charge in the lowest Landau level are known not to commute with each other.

The simplest case we have encountered in these instances is a space with constant noncommutativity:

$$[x^\mu, x^\nu] = i\Theta^{\mu\nu} \qquad (\mu, \nu = 1, 2, \cdots, d), \tag{1}$$

where $\Theta^{\mu\nu} = -\Theta_{\mu\nu}$ are real parameters, of dimension length squared. Classical field theory in such a space can be realized as a deformation of the usual field theory in an ordinary (commutative) space, by changing the product of two fields to the Moyal star product [6] defined by

$$(f * g)(x) = \exp(\frac{i}{2}\Theta^{\mu\nu}\partial^x_\mu \partial^y_\nu) f(x) g(y)|_{y=x}. \tag{2}$$

Note that the first term on the right side gives the ordinary product, recovered in the limit $\Theta_{\mu\nu} \to 0$. Also the commutatator (1) is realized as $[x^\mu, x^\nu] = x^\mu * x^\nu - x^\nu * x^\mu$.

An interesting issue of great importance for future applications of NCFT is whether spatial noncommutativity at short distances could show up in the low energy effective theory, or in the critical behavior at large distances. Conventional wisdom seems to point to a negative answer: When the distance scale under consideration is much larger than the length scale given by the coordinate noncommutativity, the effects of the latter should be negligible. More concretely, this argument goes as follows: The essential difference between the star and ordinary product is the terms other than the first one in eq. (2), which all

involve derivatives of higher orders. It is well-known in theory of critical phenomena that the interactions containing derivatives of higher orders are all irrelevant. So one would expect noncommutativity effects to vanish at sufficiently low energies.

The other side of the coin is that the commutator (1) leads to the uncertainty relation $\Delta x^\mu \Delta x^\nu \sim \Theta_{\mu\nu}$. It tells us that for a wave packet in this space, if we make its size in x^μ-direction small, then its size in x^ν-direction will become big. So the ultraviolet (UV) effects in x^μ-direction are entangled with the infrared (IR) effects in x^ν-direction. Conceptually we believe it is such UV-IR mixing [7], implied by coordinate commutation relations (1), that makes noncommutativity effects capable of showing up at large distances. This motivated us to study, using perturbative renormalization group techniques, the critical behavior of the noncommutative Landau-Ginsburg model (NCLGM). As reported below, though the upper critical dimension remains to be four, we have found a number of effects that exhibit the Θ-dependence at or near criticality. This confirms the entanglement between the UV (at the noncommutative scale, $\sqrt{\Theta}$) and the IR (at the scale of the correlation length) effects in a space described by eq. (1).

II. NONCOMMUTATIVE LANDAU-GINSBURG MODEL

For simplicity, we first consider the $d = 4$ case, with only $\Theta^{12} = \Theta^{34} = \Theta \neq 0$. The Euclidean action of the Landau-Ginsburg model with a real scalar field reads

$$S = -\int d^4x [\frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{g}{4!} \phi * \phi * \phi * \phi]. \tag{3}$$

Here the star product is defined in eq. (2). Since the star product of two fields differs from the ordinary one only by a total divergence, the quadratic terms, as well as the ϕ-propagator, in NCLGM remain the same as in the ordinary Landau-Ginsburg model (OLGM). Also for a uniform phase with $\phi = const.$, the quartic terms are the same as in OLGM. So assuming uniform phases, the phase structure at tree level is the same as in the ordinary case: The critical point is still given by $m^2 = 0$, separating the symmetric phase (with $m^2 > 0$ $\langle \phi \rangle = 0$),

and the symmetry broken phase (with $m^2 < 0$ and $\langle\phi\rangle = const \neq 0$). In this paper we consider only approaching to criticality from the symmetric phase.

The star product in the quartic term is invariant under only cyclic permutations of the factors. Using path integral, the Feynman rule in momentum space for the ϕ^4 vertex acquires an additional phase factor:

$$\exp[-(i/2)\sum_{i<j} k_i \wedge k_j], \tag{4}$$

where k_i are the cyclically ordered momenta flowing into the vertex, and $k_i \wedge k_j \equiv k_{i,\mu}\Theta^{\mu\nu}k_{j,\nu}$. To keep track of the ordering of the lines, we either allow lines crossing over each other, or introduce a double-line representation for the ϕ-propagator as if it is a marix. Either way, one can classify Feynman diagrams into planar and non-planar graphs [8]. Alternatively, to simplify the symmetry factors, one can symmetrize the phase factor (4), resulting in the momentum space form of the quartic interactions:

$$S_4 = -\frac{1}{4!}\int^{\Lambda} \phi(4)\phi(3)\phi(2)\phi(1)u(4321), \tag{5}$$

where $\int^{\Lambda} = \int_0^{\Lambda}\left(\prod_i \frac{d^4k_i}{(2\pi)^4}\right)(2\pi)^4\delta^{(4)}(\sum_i k_i)$. The function $u(4321)$ is given by

$$\begin{aligned} u(4321) = \frac{g}{3}\Big[&\cos(\frac{k_1\wedge k_2}{2})\cos(\frac{k_3\wedge k_4}{2}) \\ &+ \cos(\frac{k_1\wedge k_3}{2})\cos(\frac{k_2\wedge k_4}{2}) \\ &+ \cos(\frac{k_1\wedge k_4}{2})\cos(\frac{k_2\wedge k_3}{2})\Big]. \end{aligned} \tag{6}$$

It turned out [8] that a planar graph always gives the same contribution in OLGM (with $\Theta = 0$) multiplied by a phase factor, that depends only the external momenta and their cyclic ordering. Thus, planar graphs share the same UV divergences as in OLGM, so we still need a cut-off Λ in momentum space to regulate the NCFT. But the behavior of non-planar graphs is very different from their $\Theta = 0$ counterpart: Rapid oscillations of internal momentum dependent phase factors in the integrand make non-planar graphs in NCFT less divergent than in the ordinary theory. For example, vertex corrections at one loop in ordinary ϕ^4 theory are known to be logarithmically divergent; from this one guesses that

non-planar vertex diagrams should be finite in NCLGM. Indeed, this can be confirmed by explicit calculations (see below).

Perturbative calculations in this NCFT have been done at one [7] and two [9] loops. It was found that for the inverse propagator, non-planar graphs make the two limits, the cut-off $\Lambda \to \infty$ and the external momentum $k \to 0$, not interchangeable. If one insists to have $\Lambda \to \infty$ first, then the propagator becomes singular as $k \to 0$, seemingly obstructing a Wilsonian RG analysis [7]. We observe that this potential IR problem can be bypassed by using a shell integration in momentum space in RG analysis, as advocated by Shankar [11] and Polchinski [12].

III. RG ANALYSIS IN D DIMENSIONS

According to ref. [11], the RG transformation can be found by using (the cumulant expansion with) usual Feymann diagrams, with the loop integrals being over a thin shell $\Lambda/s \leq k \leq \Lambda$ in momentum space ($s > 1$). This shell integration represents the elimination of the fast modes $\phi_f(k)$ with momentum k within the above shell, resulting in an effective action for the slow modes $\phi_s(k)$ with $0 \leq k \leq \Lambda/s$. Then we rescale the momentum $k \to sk$ and field variables, to make the effective action to be of the same form as the original one, but with a new set of coupling constants. The relations between the new and the old coupling constants define the *RG transfomation* of the system. As usual, we will adopt dimensional regularization, in which we first finish tensor operations involving $\Theta_{\mu\nu}$ in four dimensions, then continue the loop integrals with scalar integrand to d dimensions.

Applying this procedure to the quadratic term $S_0 = -\frac{1}{2}\int^\Lambda k^2\phi(-k)\phi(k)$ and requiring it be a fixed-point action, leads to the following RG transformations

$$k' = sk, \qquad \phi'(k') = s^{-\frac{d+2}{2}}\phi_s(k). \tag{7}$$

Similarly for the term $S_2 = -\frac{m^2}{2}\int^\Lambda \phi(-k)\phi(k)$, the mode elimination by shell integration, together with RG transformation (7), leads to the tree level RG transformation:

$$m'^2 = s^2 m^2. \tag{8}$$

So the quadratic S_2 is relevant in the RG sense. So far there is no difference between NCLGM and OLGM.

To proceed, if one naively expand the cosine factors in the quartic term (6) in powers in momenta, and apply the above RG transformation to each term, then at tree level this leads to the conclusion that all the Θ dependent terms are irrelevant in any dimension. However, this expansion does not respect the star product structure, the intrinsic feature of any field theory on an NC space, that coherently organizes infinitely many higher order derivative terms. Though each of them behaves like irrelevant, their coherent sum may gives rise to non-trivial effects. Indeed, at least at one loop, it has been shown [7] that counterterms have the same star product structure. Thus, the operators allowed to appear in the Wilsonian effective action must be always of the the form of a star product with the same Θ parameter. The generic quartic terms are always of the form of eq. (6), with the prefactor g a function of momenta. Therefore, we should classify the terms in the square bracket in eq. (6) as a *marginal* operator and its marginality is protected from quantum fluctuations in NCFT intrinsically by geometry. The difference of our treatment from the RG analysis in the commutative case is that we apply the usual RG transformation only to the coefficient $g(k_1, k_2, k_3, k_4)$, with the star product structure intact, and define its relevance, irrelevance etc as usual. Thus at tree level $g' = s^{4-d} g$.

At one-loop level, quartic terms give rise to a tadpole diagram. (See Fig. 1.)

Note that the loop momentum runs only over the shell $[\Lambda/s, \Lambda]$. The tadpole diagram contributes a correction to the quadratic term

$$S_2' = -\frac{1}{2}\int_0^{\frac{\Lambda}{s}} \phi(-k)\phi(k)\Gamma_2(k), \tag{9}$$

where $\Gamma_2(k)$ is given by (for $s = 1 + t$ very close to unity)

$$\begin{aligned}\Gamma_2(k) &= \frac{g}{6}\int_{\frac{\Lambda}{s}}^{\Lambda} \frac{d^d p}{(2\pi)^d}\frac{2+\cos k\wedge p}{p^2+m^2}, \\ &= \frac{g}{6}K_d\frac{\Lambda^d t}{\Lambda^2+m^2}\left(3-\frac{1}{8}\Theta^2\Lambda^2k^2\right).\end{aligned} \tag{10}$$

where $K_d = S_d/(2\pi)^d$ and S_d is the surface area of a unit sphere in d dimensions. In the last line, we have retained only terms up to second order in momentum k, since higher-order terms in k are irrelevant.

The first term in eq. (10) is k-independent, modifying the RG transformation (8) to the following one-loop Θ-independent RG equation

$$\frac{dr}{dt} = 2r + \frac{1}{2}u(1-r)K_d, \tag{11}$$

with dimensionless $r \equiv m^2/\Lambda^2$ and $u \equiv g\Lambda^{d-4}$; $r \ll 1$ is assumed.

However, in eq. (10) the second term comes from non-planar graphs. It is both Θ- and k-dependent, and is marginal under the RG transformation (7), since *it modifies the kinetic term* S_0. Using field theory terminology, it gives rise to *wave function renormalization* for the ϕ-field, that explicitly *depends on noncommutativity.* This is our key observation in this paper. With this term, the Gaussian fixed point action S_0 is modified to

$$S_0' = -\frac{1}{2}\Big[1 - \frac{u}{48}K_d(\Theta\Lambda^2)^2 t\Big]\int_0^{\frac{\Lambda}{s}} k^2\phi(-k)\phi(k) \tag{12}$$

Using $1+\gamma t \approx s^\gamma$, and introducing a dimensionless $\theta = \Theta\Lambda^2$ for noncommutativity, we see that the ϕ-field acquires an anomalous dimension, modifying eq. (7) to

$$\phi'(k') = s^{-\frac{d+2-\gamma(u,\theta)}{2}}\phi(k) \tag{13}$$

with the one-loop γ depending on noncommutativity:

$$\gamma(u,\theta) = -\frac{1}{48}uK_d\theta^2. \tag{14}$$

This is a novel result, because in OLGM the anomalous dimension vanishes at one loop. It is a consequence of the nontrivial UV/IR mixing in NCLGM. Also unusual is that the *negative* value of this anomalous dimension. It may significantly affect the stability of the symmetric phase. We will come back to this point later.

To calculate one-loop corrections to the quartic interaction vertex, as in OLGM [10], we set all the external momenta to zero. With this prescription, a direct calculation shows that

the one-loop RG equation in NCLGM for the quartic coupling u is the same as in OLGM; i.e.,

$$\frac{du}{dt} = (4-d)u - \frac{3}{2}u^2 K_d. \tag{15}$$

With RG equations (11), (15) and the anomalous dimension (14), we can now proceed to take a closer look on the new physics due to noncommutativity.

IV. IN $D \geq 4$ DIMENSIONS

In NCLGM, the upper critical dimension remains to be $d = 4$. When $d > 4$, the quartic coupling u is irrelevant. So it is the unique, trivial Gaussian fixed point, $u^* = 0 = r^*$, that controls the IR asymptotically free low-energy behavior, with the same sets of critical exponents as usual in mean field theory: $\nu = 1/2$ and $\eta = \gamma(u^*, \theta) = 0$.

However, this is not the whole story, when we consider approaching to the critical point $r = 0$. In fact, the modified scaling law (13) gives a two-point correlation function behaving like

$$\langle \phi(x)\phi(0) \rangle \sim \frac{1}{x^{d-2+\gamma}}. \tag{16}$$

Due to the minus sign in (14), for very large noncommutative parameter θ, the above correlation function does not diverge, which signals an instability of the system. The critical value, θ_c, is given by condition

$$u\theta_c^2 = \frac{48(d-2)}{K_d}. \tag{17}$$

More precisely, the parameter space for NCLGM is now three-dimensional, described by (u, r, θ). The condition (17) gives us a surface in the parameter space. To access the Gaussian fixed point, we have to fine-tune the parameter θ to make $\theta < \theta_c(u)$. Of course, the more close to the fixed point, the less important is the condition (17), since θ_c is pushed to infinity when arriving at the Gaussian fixed point.

In the critical dimension $d = 4$, the one-loop non-zero anomalous dimension (14) is expected to modify the logarithmic corrections to the scaling laws at criticality.

V. IN $D = 4 - \varepsilon$ DIMENSIONS

If the dimension is slightly lower than four, the Gaussian fixed point becomes unstable in IR, and we have a new IR stable fixed point, the noncommutative counterpart of the Wilson-Fisher (NCWF) fixed point. Besides the noncommutativity parameter θ, its position in (r, u)-space for small $\varepsilon \equiv 4 - d$ is the same as in usual OLGM:

$$u^* = \frac{16\pi^2}{3}\varepsilon, \qquad r^* = -\frac{1}{6}\varepsilon. \tag{18}$$

At this fixed point, the critical exponent ν is unchanged: $\nu = \frac{1}{2} + \frac{\varepsilon}{12}$, but the one-loop critical exponent η becomes non-vanishing:

$$\eta = \gamma(u^*, \theta) = -\frac{\varepsilon\theta^2}{72}. \tag{19}$$

This result is characteristic of the NCLGM. We would like to stress three important aspects of the critical exponent (19): (1) It starts at order of ε, while in OLGM it starts at order of ε^2 from a higher-order calculation. (2) It is *negative*, while it is positive in OLGM. (3) It looks non-universal because of its dependence on the dimensionless noncommutativity parameter θ. However, in the present case, the NCWF fixed point had better be viewed as a line of fixed points labelled by θ. Since θ originates in the microscopic sector of the system, its appearance in the macroscopic critical exponent is a genuine manifestation of *UV/IR mixing*, namely, the fingerprint of a "high-energy" parameter in low-energy phenomena.

To see how the anomalous dimension (19) affects the stability of the NCWF fixed point, let us examine the two-point correlation function

$$\langle \phi(x)\phi(0) \rangle \sim \frac{1}{x^{2-\varepsilon(1+\theta^2/72)}}. \tag{20}$$

The criterion to maintain the stability of the NCWF fixed point is that the above correlation function should be divergent for short distances. However, now we have the additional parameter θ as a new knob to tune the system. If it is too large, the correlation function can become convergent. The critical value is given by

$$\theta_c = 12/\sqrt{\varepsilon}, \tag{21}$$

which depends only on ε. Therefore, if $\theta < \theta_c$, the NCWF fixed point is stable. On the contrary, for $\theta > \theta_c$ the NCWF fixed point will no longer be stable. This is reflected in the phase diagram, Fig. 2.

This situation is similar to previous RG analysis for one dimensional and three dimensional interacting fermion systems. There RG analysis could be used to show a similar instability for the Fermi liquid fixed point. To get the picture of the Luttinger liquid (in $1d$) and BCS superconductivity (in $3d$), one had to determine nonperturbatively the underlying physics for the new phase. Here to gain knowledge of the new phase for $\theta > \theta_c$, we also need extra efforts. But the study of the new phase in the large θ limit is beyond the scope of the present paper.

VI. CONCLUSIONS AND DISCUSSIONS

With the symmetric phase in NCLGM as an example, we have discussed the general features of critical phenomena on a noncommutative space. We have demonstrated that small spatial noncommutativity does not change the phase structure of the system, its upper critical dimension (four), nor the RG flow equations for r and u, the positions of fixed points in (r,u)-space and their nature. So the critical behavior in $d > 4$ is exactly the same as in the commutative case. On the other hand, through UV-IR mixing, noncommutativity gives rise to a non-zero anomalous dimension for the order parameter. In four dimensions it modifies logarithmic corrections to the scaling laws in the critical theory, while it changes the exponent η at the NCWF fixed point in less than four dimensions. In fact, the NCWF fixed points form a line of fixed points, where the action contains a star product of ϕ^4, labeled by the noncommutativity parameter $\theta \equiv \Theta\Lambda^2$, whose inverse is the analog of the magnetic flux per plaquette in a lattice model.

Our perturbative RG analysis starts with the assumption that the symmetric phase of NCLGM is continuously connected to that of OLGM, with $\langle\phi\rangle = 0$. Our results are

consistent with this assumption for small noncommutativity, while indicating instability towards a new phase for large θ, consistent with the phase diagram conjectured in ref. [13].

We expect that noncommutativity should also show up in the critical exponents in the symmetry-broken phase in the NCLGM. Moreover, our analysis is consistent with the proposition, made recently in refs. [13] and [14], that the NCLGM is perturbatively renormalizable. Details of our computations and generalization to the complex scalar and non-relativistic cases will be published elsewhere [15].

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FIG. 1. Tadpole diagram.

FIG. 2. Phase diagram along θ parameter.

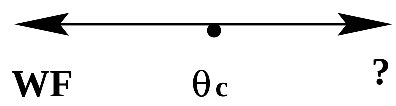

Fig.2

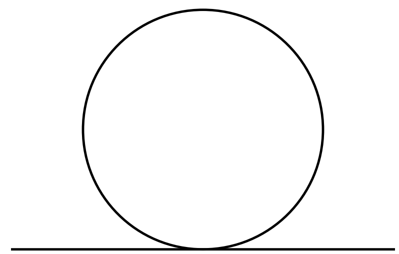

Fig. 1